A New Generalization of Rama Distribution with Application to Machinery Data

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ABSTRACT

In this study, a new model has been proposed based on the Rama distribution for modeling real-life data. The proposed model is known as Alpha Power Transformed Rama distribution. The Alpha Power Transformed model developed by Mahdavi and Kundu was used to generate the new two-parameter lifetime model. The survival function, failure rate and some statistical properties of the model are also discussed. The maximum likelihood estimation is employed for the parameter estimation of the model. The model is validated using lifetime data-sets and are compared to exponential, Erlang Truncated Exponential and Length Biased Power Lindley distribution and was found that the model provides a close fit than other competing models.

Index Terms – Alpha Power Transformation, Rama Distribution, Reliability Analysis, Moments, Entropies, Maximum Likelihood Estimation.

1. INTRODUCTION

Statistics is a tool by which we can draw inferences about random phenomena. Statistics is the scientific study of numerical data based on natural phenomena i.e. data is based on the naturally occurring events not created by the imagination. Nowadays, Statistics gain its importance because of the characteristic of its wide range of applicability in modeling data. There may be hardly any rare field where statistics has not being used. Different experimental studies result in new type of data sets and problems for which there is a need of an adequate and flexible distribution for modeling purposes. There are many existing models in the literature to fit the real data sets, but they may not be sufficient to express the behavior of statistical data sets. To overcome the deficiency of the existing models, there is a need of developing new families of probability distributions. The generalizations of existing models can provide more flexibility in modeling such types of data sets.

2. RELATED WORK

Shanker [2] introduced the Rama distribution is a parametric model that is used for modeling real-life data sets. The model has the following pdf and cdf with parameter θ.

$$g\ (y,\theta) = \left(\frac{\theta^4}{\theta^3+6}\right)(1+y^3)\,e^{-\theta y}\ ;\ y>0,\ \ \theta>0 \tag{1}$$

$$G\ (y,\theta) = 1-\left(1+\frac{\theta^3 y^3+3\theta^2 y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y};\ y>0,\ \theta>0 \tag{2}$$

Rama distribution is a mixed distribution consisting of an exponential distribution with one parameter (θ) and a gamma (4,θ) distribution. The author also discussed the properties of the model. Maximum Likelihood Estimation has been established for parameter estimation. In many statistical investigations interest lies in conducting lifetime data analysis. The modeling lifetime data analysis depends heavily on the behaviour of the hazard rate. Many lifetime models have monotone hazard rates while some are non-monotone hazard rates. Several statistical distributions exist for modeling lifetime data. The Rama distribution is used for modeling lifetime data in biomedical and engineering. The Rama distribution is specially used for lifetime data with a monotone hazard rate. However, in practice, the Rama distribution cannot be used to appropriately model with no-monotone hazard rates. The Rama distribution undergoes several generalizations. Berhane Abebe, Mussie Tesfay, et al. [3] proposed a two-parameter

Rama distribution with properties and applied it to real-life data. Rama-Kamlesh distribution with properties and applications was obtained by Shukla and Shanker [4]. Extended Rama Distribution was suggested by Alhyasat, Ibrahim et al. [5]. The model was obtained by using a concept sum of two independent random variables. In this case, both the variables follows the Rama distribution on the interval $(0,\infty)$. Generalized weighted Rama distribution was proposed by Samuel, John et al. [6] with properties and application in medical sciences. A two-parameter weighted Rama distribution was suggested by Eyob and Shanker [7] for a purpose to model data that represents a tensile strength of carbon fibers. Inverted power Rama distribution was introduced by Osuji, Samuel et al. [8]. Exponentiated Rama distribution was introduced by Kelechi et al. [9] for modeling the data that represents the tensile strength of 100 carbon fibres. The model was obtained by using the exponentiated model which was obtained by Mudholkar and Srivastava [10].Alpha power transformed Frechet distribution was introduced by Suleman et al. [11] for modeling real life data sets. The authors discussed some of the statistical properties of the distribution such as quantile function, moments, mean residual life, generating function, entropy, stochastic ordering, etc. The method of Maximum likelihood estimation is used for estimating the parameters of the distribution. Further, most of the work has been done on the alpha power transformed family of distributions namely, alpha power transformed Power Lindley was suggested by Hassan. et al.[12], alpha power transformed Lindley was introduced by Ghosh [13], alpha power transformed Inverse Power Lindley was obtained by Kumar [14], alpha power transformed Quasi Lindley suggested by Patrick and Harrison [15], alpha power transformed Weibull was proposed by Golam Kibria [16], alpha power transformed Pareto was introduced by Sakthivel, et al. [17], alpha power transformed extended exponential obtained by Hassan et al. [18], alpha power transformed Weibull was suggested by Nassar, et al. [19], Alpha power transformed Aradhana distribution was introduced by Maryam and Kannan [20] for modeling a real life time data. Some properties of the model were investigated. The maximum likelihood estimation was employed for parameter estimation. Alpha power transformed Garima distribution was proposed by Maryam and Kannan [21] with application to real-life data. Some properties of the model were investigated. These including survival function, hazard rate, moments, entropy, order statistics etc. The maximum likelihood estimation and least square estimation was investigated for the estimation of parameters, Alpha power transformed Inverse Lomax was proposed by Zein-Eldin and Elsehety [22] providing different methods for the estimation of parameters and so on.

The cumulative distribution function and probability density function of the APT family of distributions as

$$F_{APT}(y) = \begin{cases} \dfrac{\alpha^{G(y)}-1}{\alpha-1} & ;\ if\ \alpha \neq 1, \alpha > 0,\ y \in R \\ G(y) & ;\ if\ \alpha = 1, \alpha > 0,\ y \in R \end{cases} \tag{3}$$

$$f_{APT}(y) = \begin{cases} \left(\dfrac{\log\alpha}{\alpha-1}\right) g(y)\,\alpha^{G(y)} & ;\ if\ \alpha \neq 1, \alpha > 0, y \in R \\ g(y) & ;\ if\ \alpha \neq 1, \alpha > 0, y \in R \end{cases} \tag{4}$$

The purpose of this research is to implement a new generalization of the Rama distribution that addresses the shortcomings of the existing distributions in terms of lifetime data modeling. This article is organized in the following sections; in section 2, the alpha power transformed (APTR) Rama distribution is introduced. Reliability properties of the model are studied in section 3. Various statistical properties such as moments, order statistics, entropy etc. are studied in section 4. In section 5, the model parameters were investigated with the method of maximum likelihood estimation. The application of the proposed distribution is illustrated with the help of two real life-data sets in section 6. The conclusions are given in section 7 respectively.

3. PROPOSED MODEL

The probability density function of the model with two parameters θ and α, can be obtained by using (1) and (2) of the Rama distribution in the model (3), proposed by Mahdavi and Kundu [1].

$$f_{APR}(y,\theta,\alpha)=\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)(1+y^3)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}\quad ;\ if\ \alpha\neq 1,\alpha>0,y\in R \tag{5}$$

The cumulative distribution function of the proposed model as

$$F_{APR}(y,\theta,\alpha)=\frac{\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}-1}{\alpha-1}\ ;\quad if\ \ \alpha\neq 1,\alpha>0,\ y\in R \tag{6}$$

The figure 1 and figure 2 are pdf and cdf of the proposed model.

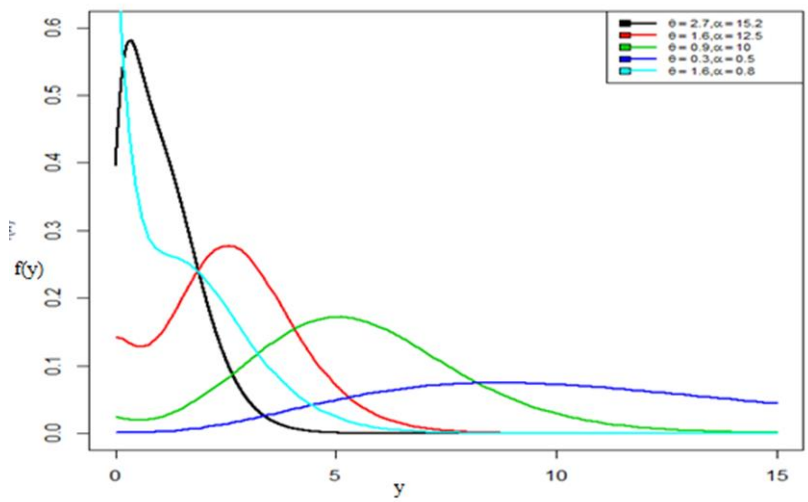

Figure 1 Pdf of Alpha Power Rama Distribution

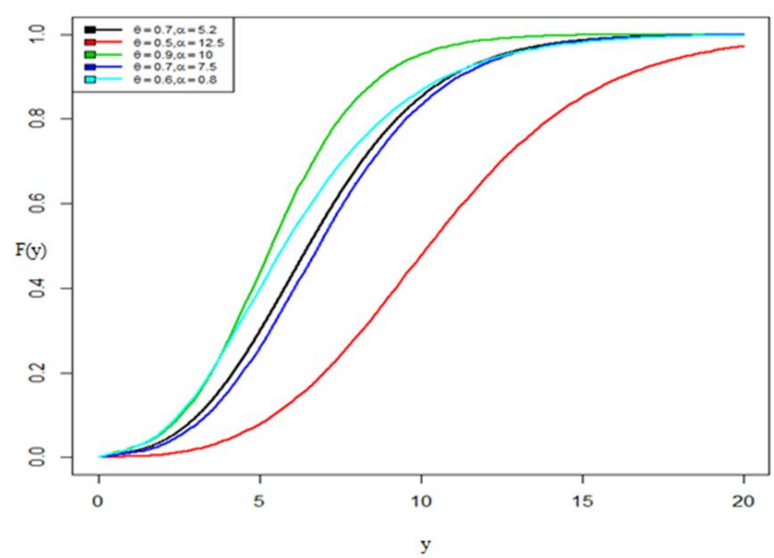

Figure 2 Cdf of Alpha Power Rama Distribution

4. RELIABILITY PROPERTIES OF THE MODEL

In this section, we discuss the survival function and failure rate of the proposed model.

The survival function or reliability function of the model is

$$S(y,\theta,\alpha)=\frac{\alpha-\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}}{\alpha-1} \tag{7}$$

The failure rate of the model is

$$h(y,\theta,\alpha)=\frac{\alpha^{-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}}{1-\alpha^{-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}}\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y^3\right)e^{-\theta y}\log\alpha \tag{8}$$

The figure 3 and figure 4 survival function and failure rate model for different values of the parameters (θ, α).

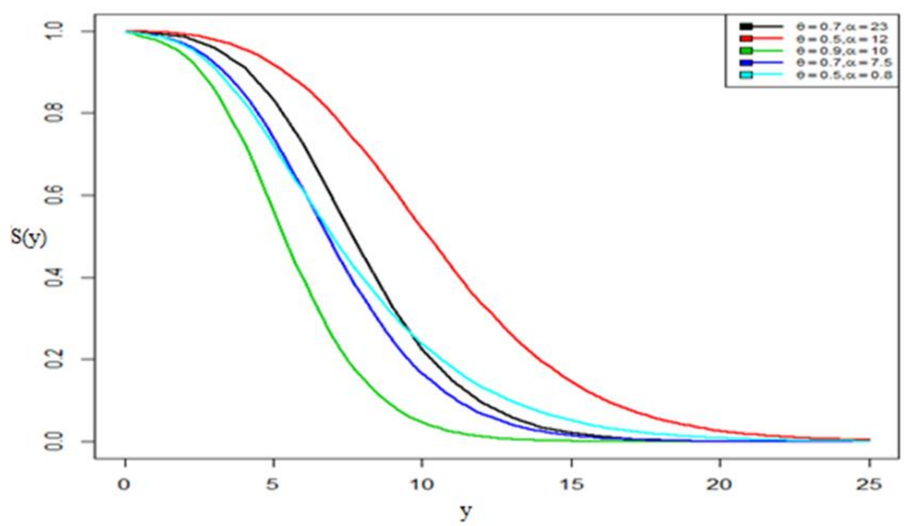

Figure 3 Survival Plot of Alpha Power Rama Distribution

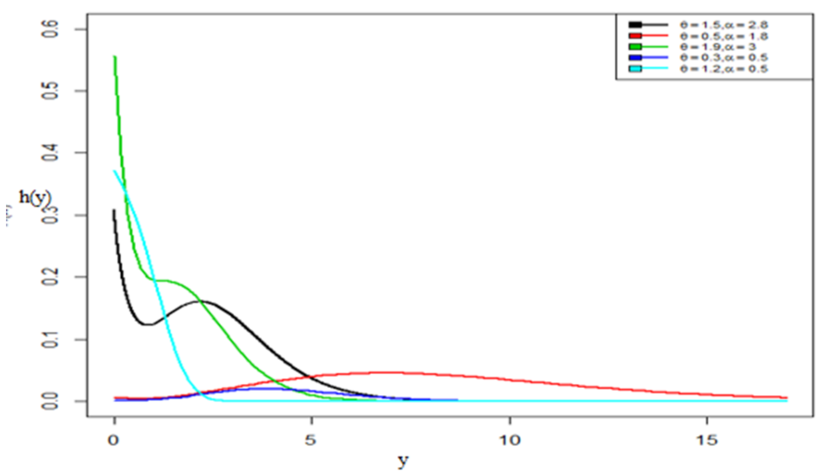

Figure 4a Failure Rate of Alpha Power Rama Distribution

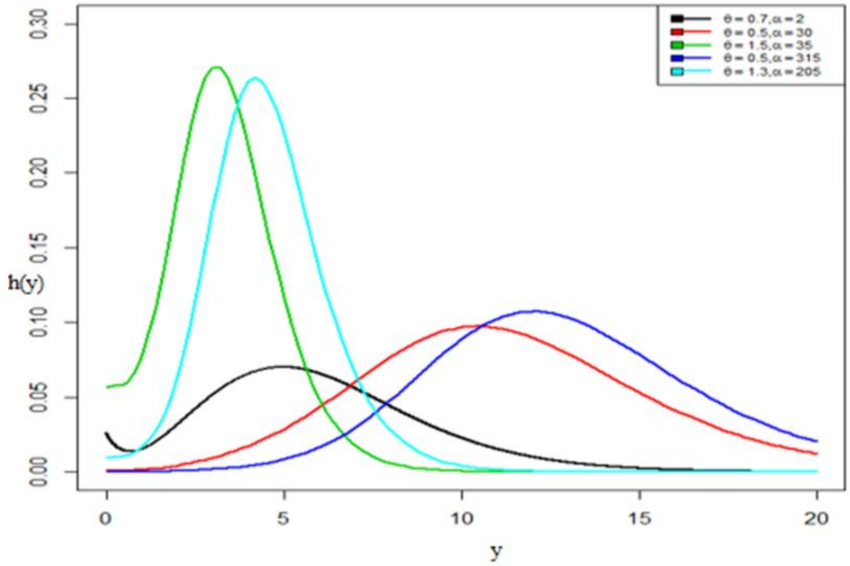

Figure 4b Failure Rate of Alpha Power Rama Distribution

From the figure 4a and 4b, the behavior of the hazard function is decreasing-increasing-decreasing and Left-Skewed for the parametric values $\theta > 1, \alpha > 1$ and for $\theta < 1, \alpha > 1$ it is Right -Skewed for increasing the value of parameter α. The hazard function is decreasing for $\theta > 1, \alpha < 1$.

5. STATISTICAL PROPERTIES OF THE MODEL

5.1 Moments

Let Y denotes the random variable follows Alpha Power Transformed Rama distribution with parameters θ, α, then the n^{th} order moment $E(Y^n)$ of Alpha Power Transformed Rama distribution can be obtained as

$$E(Y^n) = \int_0^\infty y^n \; f_{APR}(y)\, d\, y \tag{9}$$

Substituting the Equation (5) in the Equation (9), we obtain the n^{th} order moment of Alpha Power Rama distribution as

$$E(Y^n) = \int_0^\infty y^n \left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y^3\right)e^{-\theta y}\; \alpha^{1-\left(1+\frac{\theta^3 y^3+3\theta^2 y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}} \right) dy \tag{10}$$

Using the power series expansion,

$$\alpha^z = \sum_{r=0}^{\infty} \frac{(\log\alpha)^r}{r!} z^r$$

The Equation (10) becomes,

$$E(Y^n) = \sum_{r=0}^{\infty} \frac{(\log\alpha)^r}{r\,!} \left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right) \int_0^\infty y^n \left(1+y^3\right)\; e^{-\theta y} \left(1-\left(1+\frac{\theta^3 y^3+3\theta^2 y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}\right)^r d\, y \tag{11}$$

Now applying binomial expansion to equation (11),

$$(1-y)^n = \sum_{r=0}^{\infty} (-1)^r \binom{n}{r} y^r$$

Thus, the n^{th} order moment of Alpha Power Rama distribution can be obtained as

$$E(Y^n) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)\,r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}(-1)^s 3^u 2^v\left(\frac{\theta^3\,\Gamma(n+3t-u-v+1)}{\theta^n\,(s+1)^{n+3t-u-v+11}}+\frac{\Gamma(n+3t-u-v+4)}{\theta^n\,(s+1)^{n+3t-u-v+4}}\right)$$
(12)

Letting $n = 1$ in Equation (12) we get first moment of Alpha Power Rama distribution which is given by

$$E(Y) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)\,r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}(-1)^s 3^u 2^v\left(\frac{\theta^3\,\Gamma(3t-u-v+2)}{\theta(s+1)^{3t-u-v+2}}+\frac{\Gamma(3t-u-v+5)}{\theta\,(s+1)^{3t-u-v+5}}\right)$$

Second moment of the Alpha Power Rama distribution is obtained by putting the value of $n = 2$ in Equation (12) as

$$E(Y^2) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)r!}\right)(-1)^s 3^u 2^v\left(\frac{1}{\theta^3+6}\right)^{t+1}\left(\frac{\theta^3\,\Gamma(3t-u-v+3)}{\theta^2\,(s+1)^{3t-u-v+3}}+\frac{\Gamma(3t-u-v+6)}{\theta^2\,(s+1)^{3t-u-v+6}}\right)$$

Similarly, we can obtain the third, fourth and so on, moments of the distribution.

5.2 Generating Function

The moment generating function of the Alpha Power Rama distribution can be obtained as

$$M_Y(t) = E(e^{ty}) = \int_0^{\infty} e^{ty} f_{APR}(y)\,dy \qquad (13)$$

Using Taylor Series expansion to (13),

$$M_Y(t) = E(e^{ty}) = \int_0^{\infty}\left(1+ty+\frac{(ty)^2}{2!}+\dots\right) f_{APR}(y)dy$$

$$= \int_0^{\infty}\sum_{p=0}^{\infty}\frac{t^p}{p!} y^p f_{APR}(y)dy$$

$$= \sum_{p=0}^{\infty}\frac{t^p}{p!}\int_0^{\infty} y^p f_{APR}(y)dx$$

$$= \sum_{p=0}^{\infty}\mu_p{'}\frac{t^p}{p!}$$

$$M_Y(t) = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{p=0}^{\infty}\frac{t^p}{p!}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}(-1)^s 3^u 2^v\left(\frac{\theta^3\,\Gamma(p+3t-u-v+1)}{\theta^p\,(s+1)^{3t-u-v+p+1}}+\frac{\Gamma(p+3t-u-v+4)}{\theta^p\,(s+1)^{3t-u-v+p+4}}\right)$$

Similarly, the characteristic function of Alpha Power Transformed Rama distribution can be obtained as

$$\psi_Y(t) = M_Y(it)$$

$$\psi_Y(t)=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{p=0}^{\infty}\frac{(it)^p}{p!}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)\,r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}(-1)^s 3^u 2^v\left(\frac{\theta^3\,\Gamma(p+3t-u-v+1)}{\theta^p\,(s+1)^{3t-u-v+p+1}}+\frac{\Gamma(p+3t-u-v+4)}{\theta^p\,(s+1)^{3t-u-v+p+4}}\right)$$

6. ENTROPY

Entropy provides an outstanding tool to compute the amount of information or uncertainty contained in a random observation concerning its parent distribution. A large value of entropy concludes the more uncertainty in the data.

6.1 Renyi Entropy

The concept of Renyi entropy was developed by Alfred Renyi [23] to measure the randomness, diversity, uncertainty of a system. The Renyi entropy of order δ is defined by

$$R(\delta)=\frac{1}{1-\delta}\log\int_0^\infty f^\delta(y)\,dy,\quad \delta>0,\ \ \delta\neq 1 \qquad (14)$$

The renyi entropy of Alpha Power Rama distribution can be obtained by using (5) in equation (14),

$$R(\delta)=\frac{1}{1-\delta}\log\int_0^\infty\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y^3\right)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}\right)^\delta dy$$

$$R(\delta)=\frac{1}{1-\delta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^\delta\left(\frac{\theta^4}{\theta^3+6}\right)^\delta\int_0^\infty\left(1+y^3\right)^\delta e^{-\theta\delta y}\,\alpha^{\delta\left(1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}\right)}dy$$

$$R(\delta)=\frac{1}{1-\delta}\log\left(\frac{\log\alpha}{\alpha-1}\right)^\delta\left(\frac{\theta^4}{\theta^3+6}\right)^\delta\sum_{r=0}^{\infty}\frac{(\log\alpha)^r}{r!}\int_0^\infty\left(1+y^3\right)^\delta e^{-\theta\delta y}\,\delta^r\left(1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)\right)^r dy$$

On simplification, we obtain the renyi entropy of the distribution.

$$R(\delta)=\frac{1}{1-\delta}\log\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{n=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\binom{\delta}{n}\left(\frac{(-\delta\log\alpha)^r}{r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+\delta}\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^\delta(-1)^s 2^v\theta^{4\delta-u-v}3^u\,\frac{\Gamma(3t+3n-u-v+1)}{\left(\theta(\delta+s)\right)^{(3t+3n-v-u)}}$$

6.2 Tsallis Entropy

Tsallis entropy [24] for a continuous random variable Y is defined as

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\int_0^\infty f^\lambda(y)dy\right) \qquad (15)$$

The Tsallis entropy of Alpha Power Rama distribution can be obtained by using (5) in equation (15) as

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\log\alpha}{\alpha-1}\right)^\lambda\left(\frac{\theta^4}{\theta^3+6}\right)^\lambda\int_0^\infty\left(1+y^3\right)^\lambda e^{-\theta\lambda y}\,\alpha^{\lambda\left(1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}\right)}dy\right)$$

$$S_\lambda=\frac{1}{\lambda-1}\left(1-\left(\frac{\log\alpha}{\alpha-1}\right)^\lambda\left(\frac{\theta^4}{\theta^3+6}\right)^\lambda\sum_{r=0}^{\infty}\frac{(\lambda\log\alpha)^r}{r!}\int_0^\infty\left(1+y^3\right)^\lambda e^{-\theta\lambda y}\left(1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}\right)^r dy\right)$$

Similarly, on simplification we obtain the Tsallis entropy of proposed model.

$$S_\lambda = \frac{1}{\lambda-1}\left(1-\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{n=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\binom{\lambda}{n}\left(\frac{(-\lambda\log\alpha)^r}{r!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)^{\lambda}\left(\frac{1}{\theta^3+6}\right)^{t+\lambda}(-1)^s 2^v \theta^{4\lambda-u-v} 3^u \frac{\Gamma(3t+3n-u-v+1)}{(\theta(\lambda+s))^{(3t+3n-v-u)}}\right)$$

7. BONFERRONI AND LORENZ CURVE

The Bonferroni [25] and Lorenz [26] curve have wide range of applicability in various fields like economics, reliability, medicine, demography etc. The Bonferroni curve can be obtained as

$$B(p) = \frac{1}{\mu_1' p}\int_0^q y\, f(y)\, dy$$

and

$$L(p) = p\, B(p) = \frac{1}{\mu_1'}\int_0^q y\, f(y)\, dy$$

Here, we define the first raw moment as

$$\mu_1' = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\binom{u}{v}\left(\frac{(\log\alpha)^{r+1}}{(\alpha-1)\, r!}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}(-1)^s 3^u 2^v\left(\frac{\theta^2\,\Gamma(3t-u-v+2)}{(s+1)^{3t-u-v+2}}+\frac{\Gamma(3t-u-v+5)}{\theta\,(s+1)^{3t-u-v+5}}\right)$$

and $q = F^{-1}(p)$

Benforreni curve of the Alpha Power Rama distribution can be obtained as

$$B(p) = \frac{1}{\mu_1' p}\int_0^q y\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)(1+y^3)e^{-\theta y}\alpha^{1-\left(1+\frac{\theta^3 y^3+3\theta^2 y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}\, dy$$

Again, using the power series expansion,

$$\alpha^z = \sum_{r=0}^{\infty}\frac{(\log\alpha)^r}{r!}z^r$$

$$B(p) = \frac{1}{\mu_1' p}\sum_{r=0}^{\infty}\frac{(\log\alpha)^r}{r!}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\int_0^q y(1+y^3)e^{-\theta y}\left(1-\left(1+\frac{\theta^3 y^3+3\theta^2 y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}\right)^r dy$$

On simplification, we get,

$$B(p) = \frac{1}{\mu_1' p}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\left(\frac{(-\log\alpha)^i}{r!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^3+6}\right)^{t+1}\theta^{3s-t-u+4}\, 2^u 3^t\left(\frac{1}{r+1}\right)^{3s-u+2}$$

$$\left((\gamma(\theta(r+1))\, q, (3s-u+2)) + \left(\frac{1}{\theta(r+1)}\right)^3 \gamma\, ((\theta\,(r+1))q, (3s-u+5))\right)$$

Similarly, the Lorenz curve can be obtained as

$$L(p)=\frac{1}{\mu_1'}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{t=0}^{\infty}\sum_{u=0}^{\infty}\binom{r}{s}\binom{s}{t}\binom{t}{u}\left(\frac{(-\log\alpha)^r}{r\,!}\right)\left(\frac{\alpha\log\alpha}{\alpha-1}\right)\left(\frac{1}{\theta^3+6}\right)^{s+1}\theta^{3s-t-u+4}\,2^u3^t\left(\frac{1}{i+1}\right)^{3s-u+2}$$

$$\left(\left(\gamma\left(\theta(r+1)\right)q,(3s-u+2)\right)+\left(\frac{1}{\theta(r+1)}\right)^3\gamma\left(\left(\theta(r+1)\right)q,(3s-u+5)\right)\right)$$

8. ORDER STATISTICS

Let $y_1, y_2, y_3, \ldots, y_n$ be the random sample from population with pdf f_{APR} and cdf F_{APR}.

$$f_{Y(r)}(y)=\frac{n!}{(r-1)!\,(n-r)!}f_Y(y)\left[F_Y(y)\right]^{r-1}\left[1-F_Y(y)\right]^{n-r} \tag{16}$$

Inserting Equation (5) and (6) in Equation (16), the pdf of r^{th} order statistic $Y_{(r)}$ of the Alpha Power Rama distribution is given by

$$f_{Y(r)}(y)=\frac{n!}{(r-1)!\,(n-r)!}\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)(1+y^3)\,e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}$$

$$\left(\frac{\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{r-1}\left(1-\frac{\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-r}$$

Therefore, the probability density function of the higher order statistic $Y_{(n)}$ can be obtained as

$$f_{Y(n)}(y)=n\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y^3\right)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}\right)\left(\frac{\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-1}$$

The probability density function of the first order statistic $Y_{(1)}$ can be obtained as

$$f_{Y(1)}(y)=n\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y^3\right)e^{-\theta y}\,\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}\right)\left(1-\frac{\alpha^{1-\left(1+\frac{\theta^3y^3+3\theta^2y^2+6\theta y}{\theta^3+6}\right)e^{-\theta y}}-1}{\alpha-1}\right)^{n-1}$$

9. PARAMETER ESTIMATION

9.1 Maximum Likelihood Estimation

Let $y_1, y_2, y_3 \ldots\ldots y_n$ be a random sample from Alpha Power Rama distribution with parameters (α, θ), then the likelihood function of the distribution is

$$L(y,\theta,\alpha)=\prod_{i=1}^{n}\left(\left(\frac{\log\alpha}{\alpha-1}\right)\left(\frac{\theta^4}{\theta^3+6}\right)\left(1+y_i^{\,3}\right)e^{-\theta y_i}\ \alpha^{1-\left(1+\frac{\theta^3 y_i^{\,3}+3\theta^2 y_i^{\,2}+6\theta y_i}{\theta^3+6}\right)e^{-\theta y_i}}\right) \tag{17}$$

Taking logarithm on both sides to Equation (17), thus we get log likelihood function,

$$\log L = n\left(\log(\log\alpha)-\log(\alpha-1)+4\log\theta-\log\left(\theta^3+6\right)\right)+\sum_{i=1}^{n}\log\left(1+y_i^{\,3}\right)+\sum_{i=1}^{n}\left(1-\left(1+\frac{\left(\theta^3 y_i^{\,3}+3\theta^2 y_i^{\,2}+6\theta y_i\right)}{\theta^3+6}\right)e^{-\theta y_i}\right)\log\alpha$$

$$-\theta\sum_{i=1}^{n}y_i \tag{18}$$

Differentiating the Equation (18), with respect to parameters (θ, α), to get the MLE's of the parameter.

$$\frac{\partial\log L}{\partial\alpha}=\frac{n}{\alpha\log\alpha}-\frac{n}{\alpha-1}+\frac{1}{\alpha}\sum_{i=1}^{n}\left(1-\left(1+\frac{\left(\theta^3 y_i^{\,3}+3\theta^2 y_i^{\,2}+6\theta y_i\right)}{\theta^3+6}\right)e^{-\theta y_i}\right)=0 \tag{19}$$

$$\frac{\partial\log L}{\partial\theta}=\frac{4n}{\theta}-\frac{3\theta^2 n}{\left(\theta^3+6\right)}-\sum_{i=1}^{n}y_i+\sum_{i=1}^{n}\left(1-\left(1+\frac{\left(\theta^3 y_i^{\,3}+3\theta^2 y_i^{\,2}+6\theta y_i\right)}{\theta^3+6}\right)e^{-\theta y_i}\right)\log\alpha=0 \tag{20}$$

The system of nonlinear equations is extremely difficult to solve algebraically due to the complicated form of the equations (19) and equation (20). As a result, we estimate the parameters using R software [27].

9.2 Simulation Study

In this section, we study the performance of ML estimators for different sample sizes (n =100, 200, 300,400, 500) and parameter *set I (2. 2, 1.4), set II (2.6, 0.2), set III (0.8, 1.8)*. We have employed the inverse cdf technique for data simulation for Alpha power Rama distribution using R software. The process was repeated 500 times for the calculation of bias. Variance and MSE. For different values of parameters of Alpha power Rama distribution, the decreasing trend is being observed in average bias, variance and MSE as we increase the sample size. Hence the performance of ML estimators is quoted well, consistent in the case of Alpha power Rama distribution.

Sample size n	**Bias (α)**	**Variance (α)**	**MSE (α)**	**Bias (θ)**	**Variance (θ)**	**MSE(θ)**
100	0.269375	2.706793	2.779356	-0.01073	0.02708	0.027198
200	-0.04248	0.943338	0.945143	0.005452	0.00451	0.004538
300	0.061723	2.027454	2.031264	-0.03224	0.01351	0.014544
400	-0.00876	0.967741	0.967817	-0.01624	0.00479	0.005058
500	-0.09409	0.714873	0.723725	-0.02747	.006281	0.007035
100	0.334714	1.970493	2.082527	0.002665	0.00027	0.000274
200	0.258713	2.337195	2.404127	-0.00115	0.00031	0.000309

300	-0.21349	2.650690	2.696267	-0.00603	0.00028	0.000312
400	-0.12148	1.740648	1.755407	-0.00402	0.00013	0.000148
500	0.219093	0.879471	0.927472	0.002039	0.00006	0.000068
100	0.502717	0.408603	0.661328	0.097892	0.04659	0.056169
200	0.214119	0.265194	0.311041	0.000847	0.01556	0.015563
300	-0.00949	0.235854	0.235944	-0.02095	0.01828	0.018714
400	-0.08384	0.065447	0.072476	-0.04197	0.00711	0.008873
500	-0.05378	0.144299	0.147191	-0.04728	0.02256	0.024798

Table 1: Average Bias, Variance. ML Estimates of Different Sample Size

10. ILLUSTRATION FOR VALIDATION OF ALPHA POWER TRANSFORMED RAMA DISTRIBUTION

In this section, the Alpha Power Transformed Rama distribution is validated by fitting the real lifetime data sets to it and the model is compared to Erlang Truncated Exponential Distribution, Exponential and Length Biased Power Lindley Distribution.

Data Set 1

The first data was studied by Bader and Priest [28] is the tensile strength of 69 carbon fibres under tension at gauge lengths of 20mm.

1.314, 1.479, 1.552,1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.14, 2.179, 2.224, 2.240, 2.253, 2.274, 2.301, 2.301, 3.433, 3.585, 3.585, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511,2.514, 2.535, 2.554, 1.312, 2.566, 2.57, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.270, 2.272, 2.770,2.773, 2.800, 1.700, 1.803, 2.809, 2.818, 2.821, 2.848, 2.88, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128,3.233.

Data Set 2

Smith and Naylor [29], Bourguignon et al. [30] took the data set and fitted it to the Weibull G family. The data collection reflects the power of 63 glass fibres with a diameter of 1.5 cm. The data set represents the strength of 63 of 1.5 cm glass fiber.

0.55, 0.74, 0.77, 0.81, 0.84, 0.93, 1.04, 1.11, 1.13, 1.24, 1.25, 1.27, 1.28, 1.29, 1.30, 1.36, 1.39, 1.42, 1.48, 1.48, 1.49, 1.49, 1.50, 1.50, 1.51, 1.52, 1.53, 1.54, 1.55, 1.55, 1.58, 1.59, 1.60, 1.61, 1.61, 1.61, 1.61, 1.62, 1.62, 1.63, 1.64, 1.66, 1.66, 1.66, 1.67, 1.68, 1.68, 1.69, 1.70, 1.70, 1.73, 1.76, 1.76, 1.77, 1.78, 1.81, 1.82, 1.84, 1.84, 1.89, 2.00, 2.01, 2.24.

The validation of the distributions are compared by using the model selection criteria such as AIC (Akaike Information Criterion), AICC (Corrected Akaike Information Criterion) and BIC (Bayesian Information Criterion). The better distribution corresponds to lesser AIC, AICC and BIC values.

$$AIC = 2p - 2\log L, \quad BIC = p\log n - 2\log L, \quad AICC = AIC + \frac{2p(p+1)}{(n-p-1)}$$

Where, ‘p’ is the number of parameters in the statistical model, ‘n’ is the sample size and L is the maximized value of the log-likelihood function.

Data sets	Distribution	ML Estimates		AIC	BIC	AICC	-2 log L
1	**APRD**	**2.4772**	**7.9962**	**123.929**	**128.3973**	**124.2983**	**124.7196**
	Exponential	0.4079	-	263.7352	265.9693	263.7900	261.7352
	Erlang TED	0.9038	0.6004	265.7352	270.2034	266.1045	261.7362
	LBPLD	1.0981	0.9701	202.2437	206.7119	202.613	198.2440
2	**APRD**	**3.3578**	**1.3350**	**73.18223**	**77.4685**	**73.58901**	**74.39958**
	Exponential	0.6636	-	179.6606	181.8038	263.7900	177.6606
	Erlang TED	1.1713	0.8360	181.6606	185.9469	182.0674	180.3918
	LBPLD	1.6062	1.4658	109.7942	114.0804	110.2009	109.9374

Table 2: Goodness of fit criteria

As compared to other distributions, the Alpha Power Transformed Rama distribution has the lowest AIC, BIC, and AICC values. As a result, we can infer that the Alpha Power Rama distribution suits the data better than the Exponential, Erlang Truncated Exponential distribution and Length biased power Lindley distribution.

11. CONCLUSION

The authors of this paper proposed a new generalization of the Rama distribution based on the APT model provided by Mahdavi and Kundu. The model is referred to as Alpha Power Transformed Rama distribution. The authors investigate some of the properties of the proposed distribution. The validity of the Alpha Power Transformed Rama distribution is demonstrated by fitting real-life data sets, and it is found that the Alpha Power Transformed Rama distribution suits lifetime data better than the Erlang Truncated Exponential Distribution, Exponential, and Length Biased Power Lindley Distribution.

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